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THE NILPOTENCY CLASS OF THE UNIT GROUP OF A MODULAR GROUP ALGEBRA I

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ABSTRACT

The nilpotency class of the unit group U of a modular p-group algebra FG is determined when p is odd and G has a cyclic commutator subgroup. This is done via an extension of a theorem of Coleman and Passman, dealing with wreath products obtained as sections of U.

1. Introduction

Let G be a finite p-group, and let F be the field with p elements. Denote by $\Delta(G)$ the augmentation ideal of the modular group algebra FG. Then the unit group of FG (which coincides with $FG \setminus \Delta(G)$) is isomorphic to the direct product $F^* \times U(G)$, where $U(G) = 1 + \Delta(G)$ is the group of normalized units. Since $U(G)$ is a finite p-group (of order $p^{|G|-1}$), it is clearly nilpotent. Its nilpotency class, denoted by cl $U(G)$, is the same as that of the whole unit group of FG. Our goal here is to study cl $U(G)$. For more extensive background see [Sh2].

Our starting point is an interesting result of Coleman and Passman [CP], showing that, if G is non-abelian, then the wreath product $C_p \,\mathrm{wr}\, C_p$ is involved in $U(G)$ (i.e. obtained as a section H/L, where $L \lhd H \leqq U(G)$). This immediately yields cl $U(G) \geqq p$ for non-abelian p-groups. Dealing with some improvements of [CP], Baginski has recently shown that cl $U(G) = p$ if $|G'| = p$ [Ba].

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In order to obtain results for a wider family of groups, it seemed natural to look for appropriate extensions of the Coleman–Passman theorem, showing that, under certain assumptions, larger wreath products are involved in $U(G)$. Our main result of this type is

THEOREM A. *Let G be a p-group ($p > 2$) with a cyclic commutator subgroup. Then the wreath product C_p wr G' is involved in $U(G)$.*

Since $\mathrm{cl}(C_p \mathrm{wr}\, G') = |G'|$ for G' cyclic [Bu], one immediately concludes that $\mathrm{cl}\, U(G) \geqq |G'|$ in this case. On the other hand, by [GL], $\mathrm{cl}\, U(G)$ is always bounded above by the nilptency class of FG, viewed as a Lie algebra. Luckily enough, this nilpotency class cannot be larger than the order of G' (apply [Sh1, cor.B1] with $e' = n'$; see also [Sh2, thm.B]). Putting everything together, we get

THEOREM B. *Let G be a p-group ($p > 2$) with a cyclic commutator subgroup. Then* $\mathrm{cl}\, U(G) = |G'|$.

A classification of all the 2-generated finite p-groups with a cyclic commutator subgroup may be found in [Mi]. These include, of course, all the metacyclic p-groups.

Theorem B may be regarded as a generalization of the above mentioned result of Baginski, dealing with the case $|G'| = p$. It serves as an important tool in [MS], where Avinoam Mann and I prove a conjecture of Sandling [Sa], showing that $\mathrm{cl}\, U(G) = p$ implies $|G'| = p$ (so that the two conditions are in fact equivalent).

The rest of the paper is organized as follows: in Section 2 we use the action of G on an appropriate subgroup H to construct certain semi-direct products in $U(G)$. This general construction is then applied in Section 3, where Theorem A is proved. As another application we show that, if G is a p-group of class two ($p > 2$), and $H < G$ is any cyclic subgroup, then $U(G)$ involves the wreath product $C_p \mathrm{wr}\, (G/N_G(H))$ (see Proposition D).

Our notation is standard. We use $\gamma_j = \gamma_j(G)$ for the lower central series, and G^{p^i} for the subgroup generated by the p^i-powers. For a subset $S \subseteq G$, denote $\hat{S} = \Sigma_{g \in S}\, g$ ($\in FG$). For an element $x = \Sigma\, k_g g \in FG$ (where $k_g \in F$), define $\mathrm{supp}(x) = \{g \in G : k_g \neq 0\}$ and $\mathrm{tr}(x) = k_1$.

2. A general construction

Our main tool is the following result, which may be viewed as an extension of the method introduced in [CP].

PROPOSITION C. *Let G be a p-group, let $H < G$, and let H_i $(0 \leqq i < m)$ be all its conjugates in G. Suppose $m > 1$ and*

(1) $H_i \cap H_j = 1$ *implies* $H_i H_j = H_j H_i$,

(2) $H_i \not\subseteq \bigcup_{j \neq i} H_j$,

(3) $|H_{i_1} H_{i_2} \cdots H_{i_p}| < |H|^p$ *for all* $i_1, \ldots, i_p$.

Put $x_i = 1 + \hat{H}_i$, $X = \langle x_i : 0 \leqq i < m \rangle$ $(\subseteq U(G))$. *Then*

(a) *X is elementary abelian with basis* $\{x_i : 0 \leqq i < m\}$.

(b) $X \cap G = 1$.

(c) *$U(G)$ contains the semi-direct product* $X \rtimes G$.

REMARK. It is clear that these conclusions remain valid in the case $m = 1$, provided $p > 2$. This will be used freely throughout the paper.

We need the following lemmas.

LEMMA C1. *Let x_i $(0 \leqq i < m)$ be commuting elements of order p in $U(G)$, on which G acts transitively. Put* $X = \langle x_i : 0 \leqq i < m \rangle$. *Then*

(a) *The x_i's form a basis for X if and only if* $\Pi x_i \neq 1$.

(b) *Suppose that the x_i's form a basis for X. Then $X \cap G = 1$ if and only if* $\Pi x_i \notin G$.

PROOF. (a) The 'only if' part is clear, so let us prove the other part. Let Y be a vector space over F with basis $\{y_i : 0 \leqq i < m\}$. Define an action of G on Y by $y_i g = y_j$ if $x_i^g = x_j$. Then Y becomes a G-module, and G acts transitively on its basis $\{y_i\}$. This implies that, if $y = \Sigma\, k_i y_i$ is a fixed point under G, then $k_i = k_j$ for all i, j, so that $y \in \langle \Sigma\, y_i \rangle$. Therefore $C_Y(G) = \langle \Sigma\, y_i \rangle$. Since this subspace is one-dimensional, and every non-zero submodule of Y possesses non-trivial fixed points, it follows that $\Sigma\, y_i$ lies in every such submodule.

Now, viewing X as a G-module, consider the linear map $f\colon Y \to X$, sending y_i to x_i $(0 \leqq i < m)$. It is clearly a G-module-homomorphism. Let $Z \subseteq Y$ be its kernel. Recall that, by our assumption (written additively), $f(\Sigma\, y_i) = \Sigma\, x_i \neq 0$. Therefore $\Sigma\, y_i \notin Z$. Applying the previous observation we conclude that $Z = 0$, so that f is an isomomorphism, and the x_i's are indeed a basis for X.

(b) Again, only the 'if' part should be established.

Let $Z = X \cap G$ (in $U(G)$), viewed as a submodule of X. Since $\Pi x_i \notin G$, it follows that, in additive notation, $\Sigma\, x_i \notin Z$. Reasoning as in part (a) (with X

instead of Y), we conclude that $Z = 0$, so that $X \cap G = 1$, as required. □

Our next question is: how can one verify that $\Pi x_i \neq 1$? In certain situations the following lemma supplies us with a simple criterion.

LEMMA C2. *Let R be a commutative algebra over F, and let y_i $(0 \leqq i < m)$ be elements of R, satisfying $y_i^2 = 0$. Assume also that $I^p = 0$, where I is the ideal generated by the y_i's. Define $x_i = 1 + y_i$ $(0 \leqq i < m)$. Then $\Pi x_i = 1$ if and only if $\Sigma y_i = 0$.*

PROOF. Put

$$S_n = \sum_{0 \leqq i_1 < \cdots < i_n < m} y_{i_1} \cdots y_{i_n} \qquad (n \geqq 1),$$

and $S_0 = 1$. Observe that $S_n = 0$ for $n \geqq p$ (since $I^p = 0$), whereas, for $n < p$, we have $S_n = S_1^n / n!$. Therefore

$$\Pi x_i = \Pi(1 + y_i) = \sum_{0 \leqq n} S_n = \sum_{0 \leqq n < p} S_n = \sum_{0 \leqq n < p} S_1^n / n! = 1 + S_1 \cdot u,$$

where $u = \Sigma_{1 \leqq n < p} S_1^{n-1} / n!$. Note that u is invertible (since $u - 1 \in I$ is nilpotent). Therefore

$$\Pi x_i = 1 \Leftrightarrow 1 + S_1 \cdot u = 1 \Leftrightarrow S_1 \cdot u = 0 \Leftrightarrow S_1 = 0.$$

Since $S_1 = \Sigma y_i$, the proof is completed. □

REMARK. If R is an algebra over a field of characteristic zero, then the condition $I^p = 0$ may be omitted. Indeed, using the exponent map (which is well defined here), one obtains $\Pi x_i = \Pi \exp(y_i) = \exp(\Sigma y_i)$, from which the result easily follows. In characteristic p the condition $I^p = 0$ enables one to use the p-version of the exponent map, $e_p(y) = \Sigma_{0 \leqq n < p} y^n / n!$.

PROOF OF PROPOSITION C. (a) Let us show first that X is elementary abelian. Note that

$$\hat{H}_i \hat{H}_j = \begin{cases} 0, & H_i \cap H_j \neq 1, \\ \widehat{H_i H_j}, & H_i \cap H_j = 1. \end{cases}$$

Therefore condition (1) implies $\hat{H}_i \hat{H}_j = \hat{H}_j \hat{H}_i$, so that the x_i's commute, and X is abelian. Now, $x_i^p = (1 + \hat{H}_i)^p = 1 + \hat{H}_i^p = 1$ (as $\hat{H}_i^2 = 0$). Therefore X has exponent p.

We have to show that the x_i's form a basis for X. Regarding Lemma C1(a), it is sufficient to prove that $\Pi x_i \neq 1$. Define $y_i = \hat{H}_i\,(0 \leqq i < m)$, and let R be the F-subalgebra generated by the y_i's in FG. Obviously, R is commutative and $y_i^2 = 0$. Moreover, condition (3) in Proposition C insures that $y_{i_1} \cdots y_{i_p} = 0$ for all $i_1, \ldots, i_p$, so that all the conditions of Lemma C2 are fulfilled. We conclude that, in order to show that $\Pi x_i \neq 1$, it is sufficient to verify that $\Sigma\, y_i \neq 0$. Apply condition (2) in order to find an element $h \in H_0 \setminus \bigcup_{j \neq 0} H_j$. Clearly, $h \in \operatorname{supp}\{\Sigma\, y_i\}$, so that $\Sigma\, y_i$ cannot vanish.

(b) By Lemma Cl(b) it suffices to show that $\Pi x_i \notin G$. Recall that $m > 1$. Write

$$\Pi x_i = \Pi(1 + y_i) = \sum_{0 \leqq n < p} S_n$$

using previous notation. We claim that $\operatorname{tr}(S_n) = 0$ for all $1 \leqq n < p$. Indeed, define

$$T_n = \{\{i_1, \ldots, i_n\} : \hat{H}_{i_1} \cdots \hat{H}_{i_n} \neq 0\}$$

(note that the product $\hat{H}_{i_1} \cdots \hat{H}_{i_n}$ is independent of the order of the factors).

The action of G on the indices $\{0, \ldots, m-1\}$ (via $ig = j$ if $H_i^g = H_j$) induces an action on T_n, by $\{i_1, \ldots, i_n\}g = \{i_1 g, \ldots, i_n g\}$. If $n < p\,(\leqq m)$, it is possible to find $g \in G$ sending i_1 to some $j \notin \{i_1, \ldots, i_n\}$. This shows that the action of G on T_n is fixed-point-free. It follows now that p divides $|T_n|$. Note that, if $\{i_1, \ldots, i_n\} \in T_n$, then

$$\operatorname{tr}(\hat{H}_{i_1} \cdots \hat{H}_{i_n}) = \operatorname{tr}(\widehat{H_{i_1} \cdots H_{i_n}}) = 1.$$

We conclude that, for $1 \leqq n < p$,

$$\operatorname{tr}(S_n) = \sum_{\{i_1, \ldots, i_n\} \in T_n} \operatorname{tr}(\hat{H}_{i_1} \cdots \hat{H}_{i_n}) = |T_n| = 0 \qquad (\text{in } F)$$

as asserted.

Finally,

$$\operatorname{tr}(\Pi x_i) = \operatorname{tr}\left(1 + \sum_{1 \leqq n < p} S_n\right) = 1 + \sum_{1 \leqq n < p} \operatorname{tr}(S_n) = 1.$$

Therefore $1 \in \operatorname{supp}(\Pi x_i)$. But we have already shown that $\Pi x_i \neq 1$. It follows that $\Pi x_i \notin G$, as required.

(c) Follows from the above. □

REMARKS. (1) *If* G is of class 2, then condition (1) in Proposition C is clearly satisfied.

(2) If H is cyclic, then condition (2) in Proposition C is satisfied. Indeed, if

$i \neq j$ then $H_i \cap H_j$ is contained in H_i^p, so that $H_i \cap (\bigcup_{j \neq i} H_j) \subseteq H_i^p$. Thus H is not covered by the union of its proper conjugates.

Let us now examine more closely the situation described above. Suppose G is a p-group of class 2, and let $H < G$ be a cyclic subgroup. Then conditions (1) and (2) in Proposition C are satisfied. Although condition (3) need not hold, we claim that it may be omitted, provided p is odd. Recall that we have only used it to show that $\Pi x_i \neq 1$, by applying Lemma C2. So let us prove directly that, in our circumstances, $\Pi x_i \neq 1$.

Using previous notation we have

$$\Pi x_i = 1 + \sum_{n \geqq 1} S_n.$$

Recall that $S_1 = \Sigma \hat{H}_i \neq 0$. Therefore, in order to obtain $\Pi x_i \neq 1$, it suffices to show that all the other summands vanish.

So fix $n \geqq 2$, and let us show that $S_n = 0$. As before

$$S_n = \sum_{\{i_1,\ldots,i_n\}\in T_n} \widehat{H_{i_1}\cdots H_{i_n}}.$$

For a subgroup $L \leqq G$, put

$$T_{n,L} = \{\{i_1, \ldots, i_n\} \in T_n : H_{i_1}\cdots H_{i_n} = L\}.$$

Clearly

$$S_n = \sum_{L \leqq G} |T_{n,L}| \cdot \hat{L}.$$

Therefore, if we show that p divides $|T_{n,L}|$ for all L, the proof will be completed. So suppose $L \leqq G$ with $T_{n,L} \neq \varnothing$. Recall that G acts on T_n by $\{i_1, \ldots, i_n\}g = \{i_1 g, \ldots, i_n g\}$, and observe that $T_{n,L}$ is $N_G(L)$-invariant with respect to this action. We will show that $N_G(L)$ acts fixed-point-freely on $T_{n,L}$.

First observe that, since G is of class 2, $N_G(H_i) = N_G(H) \lhd G$ for all i. Now, let $\{i_1, \ldots, i_n\} \in T_{n,L}$. Then $\hat{H}_{i_1}\cdots\hat{H}_{i_n} \neq 0$, $H_{i_1}\cdots H_{i_n} = L$, and $N_G(H) \subseteq N_G(L)$. We claim that this is a proper inclusion. Set $H_{i_j} = H^{g_j}$ $(j = 1, \ldots, n)$. Replacing H with H_{i_1}, we may assume $g_1 = 1$. This implies that $g_2, \ldots, g_n \notin N_G(H)$ (otherwise $\hat{H}$ would appear more than once in $\hat{H}_{i_1}\cdots\hat{H}_{i_n}$, so that $\hat{H}_{i_1}\cdots\hat{H}_{i_n} = 0$, a contradiction). Put $H = \langle h \rangle$. Then $L = \langle h, h^{g_2}, \ldots, h^{g_n} \rangle = \langle h, [h, g_2], \ldots, [h, g_n] \rangle$ and each of the $[h, g_j]$'s is central. Hence, for $2 \leqq j \leqq n$ we obtain $L^{g_j} = \langle h^{g_j}, [h, g_2], \ldots, [h, g_n] \rangle \subseteq L$, so that $g_2, \ldots, g_n \in N_G(L)$. This shows that $N_G(L) \supsetneq N_G(H)$. Now, take an element $g \in N_G(L) \setminus N_G(H)$, satisfying $g^p \in N_G(H)$. Then g acts on the conjugates of H as a product of disjoint cycles of length p. Therefore, if $\{i_1, \ldots, i_n\}$ is fixed under g, then $n \geqq p$, and, without loss of generality, $(i_1, \ldots, i_p)$ is a g-cycle. This means that the g_j's may be chosen such that $g_j = g^{j-1}$ for $1 \leqq j \leqq p$. In particular

$$H_{i_3} = H^{g^2} = \langle h^{g^2} \rangle \subseteq \langle h, [h, g^2] \rangle = \langle h, [h, g]^2 \rangle = \langle h, [h, g] \rangle = H \cdot H^g = H_{i_1} H_{i_2}$$

(recall that $p > 2$). This implies that $\hat{H}_{i_1} \hat{H}_{i_2} \hat{H}_{i_3} = 0$, a contradiction.

We conclude that g acts fixed-point-freely on $T_{n,L}$, from which it follows that $p \mid |T_{n,L}|$, so that $\Pi x_i \neq 1$. Therefore the conclusion of Proposition C is valid for H and G.

So consider the semi-direct product $W = XG = X \rtimes G \leqq U(G)$. Observe that $C_G(X) = N_G(H) \lhd G$. Therefore $N_G(H) \lhd W$, and $W/N_G(H) \cong X \rtimes (G/N_G(H))$ is isomorphic to the wreath product $C_p \operatorname{wr}(G/N_G(H))$. We have established the following

Proposition D. *Let G be a p-group ($p > 2$) of class two, and let $H < G$ be any cyclic subgroup. Then the wreath product $C_p \operatorname{wr}(G/N_G(H))$ is involved in $U(G)$.* □

It is known that, for a finite p-group P, the class of $C_p \operatorname{wr} P$ is exactly $t(P)$, the nilpotency index of the augmentation ideal $\Delta(P)$ [Bu]. Therefore we obtain

Corollary D1. *Under the above conditions* $\operatorname{cl} U(G) \geqq t(G/N_G(H))$. □

3. Proof of main result

In this section we apply Proposition C in order to prove Theorem A (from which Theorem B was shown to follow).

So let us assume, from now on, that G is a finite p-group ($p > 2$) with a cyclic commutator subgroup of order p^a.

Lemma A1. *Suppose $G' = \langle [x, y] \rangle$. Then $[x, y^{p^{a-1}}] = [x, y]^{p^{a-1}}$.*

Proof. By Hall's collection formula [HB, p. 240] we have

$$[x, y^{p^{a-1}}] \equiv [x, y]^{p^{a-1}} \bmod (H')^{p^{a-1}} \prod_{1 \leqq r < a} (\gamma_{p^r}(H))^{p^{a-1-r}}$$

where $H = \langle y, [x, y] \rangle$. We want to show that we actually have equality.

Clearly, $H' \subseteq \gamma_3(G) \subseteq (G')^p$, so that $(H')^{p^{a-1}} \subseteq ((G')^p)^{p^{a-1}} = (G')^{p^a} = 1$. Now, if $1 \leqq r < a$, then $\gamma_{p^r}(H) \subseteq \gamma_{p^r+1}(G) \subseteq \gamma_{r+3}(G)$ (since $p > 2$), while

$$\gamma_{r+3}(G) = [G', \underbrace{G, \ldots, G}_{r+1}] \subseteq (G')^{p^{r+1}}$$

(since G' is cyclic). Therefore

$$(\gamma_{p^r}(H))^{p^{a-1-r}} \subseteq ((G')^{p^{r+1}})^{p^{a-1-r}} = (G')^{p^a} = 1.$$

This completes the proof. □

Remark. It is clear that we actually get $[x, y^{p^b}] = [x^{p^b}, y] = [x, y]^{p^b}$ for any integer $b \geqq a - 1$. This will be used in the sequel.

Lemma A2. *There exist $x, y \in G$ with $G' = \langle [x, y] \rangle$, such that the subgroups $\langle x \rangle^{y^i}$ ($0 \leqq i < p^a$) are all distinct.*

Proof. The proof is by induction on $\min\{o(x), o(y)\}$, denoted by p^b. Obviously, $b \geqq a$, by the previous remark. So let us first assume that $p^b = p^a = o(x)$. It is sufficient to verify that $\langle x \rangle^{y^{p^{a-1}}} \neq \langle x \rangle$. Otherwise, $x^{y^{p^{a-1}}} = x^k$, for some k, so that Lemma A1 yields

$$x^{k-1} = [x, y^{p^{a-1}}] = [x, y]^{p^{a-1}} = [x^{p^{a-1}}, y].$$

Now, $[x, y]^{p^{a-1}} \neq 1$ (by the definition of a). Therefore $p^a \nmid k - 1$. It follows that

$$x^{p^{a-1}} \in \langle x^{k-1} \rangle = \langle [x^{p^{a-1}}, y] \rangle.$$

This contradicts the nilpotency of G.

Suppose now that $b > a$. If either $\langle x \rangle^{y^{p^{a-1}}} \neq \langle x \rangle$ or $\langle y \rangle^{x^{p^{b-1}}} \neq \langle y \rangle$ we are done, so let us assume that in both cases we have equality. This means that, for some integers k and l, we have

$$x^{y^{p^{a-1}}} = x^k, \qquad y^{p^{a-1}} = y^l.$$

It follows that

$$x^{k-1} = [x, y^{p^{a-1}}] = [x, y]^{p^{a-1}} = [x^{p^{a-1}}, y] = [y, x^{p^{a-1}}]^{-1} = y^{1-l}.$$

Let p^c be the maximal pth-power dividing both $k - 1$ and $l - 1$. Observe that, since $x^{k-1}, y^{l-1} \neq 1$, we must have $c < b$. Suppose, without loss of generality, that $p^c \,||\, k - 1$. It follows that, for some j,

$$x^{p^c} = (y^j)^{p^c}.$$

Since G' is cyclic and p is odd, G is a regular p-group [Hu, p. 322]. This implies that, defining $x_1 = x \cdot y^{-j}$, we obtain $x_1^{p^c} = 1$ [Hu, p. 326]. Observe that

$$[x_1, y] = [xy^{-j}, y] = [x, y]^{y^{-j}}.$$

We conclude that $G' = \langle [x_1, y] \rangle$, where $\min\{o(x_1), o(y)\} = p^c < p^b = \min\{o(x), o(y)\}$, so we are done by induction hypothesis. □

PROOF OF THEOREM A. Let x, y be as in Lemma A2, and set $H = \langle x \rangle$, $m = p^a$. For $0 \leqq i < m$ put $H_i = H^{y^i}$. Then the H_i's are all distinct, and are in fact all the conjugates of H in G (since $|G/N_G(\langle x \rangle)| \leqq |G/C_G(x)| \leqq |G'| = m$). We claim that conditions (1)–(3) in Proposition C are satisfied.

First, condition (2) follows from the fact that H is cyclic. Next, define $L = \langle x, x^y \rangle$. Then $L = \langle x, G' \rangle$ is normal in G. Therefore $H_i \subseteq L$ for all i. Observe that, by the remark following Lemma A1, $[x^{p^a}, y] = [x, y]^{p^a} = 1$. Therefore $x^{p^a} \in H_i$ for all i.

Two cases should be considered.

Case 1. $x^{p^a} \neq 1$.

Then $H_i \cap H_j \neq 1$ for all i, j, from which conditions (1) and (3) immediately follow.

Case 2. $x^{p^a} = 1$.

This implies $|L| \leqq |G'| \, |\langle x \rangle| = p^{2a}$. But then $H_i \cap H_j = 1$ yields $H_i H_j = L = H_j H_i$ by order considerations, so condition (1) is satisfied. For condition (3), note that $H_{i_1} \cdots H_{i_p} \subseteq L$ for all $i_1, \ldots, i_p$, which implies

$$|H_{i_1} \cdots H_{i_p}| \leqq p^{2a} = |H|^2 < |H|^p$$

(recall that p is odd).

At this stage Proposition C may be applied. We conclude that the elements $x_i = 1 + \hat{H}_i$ $(0 \leqq i < m)$ generate in $U(G)$ an elementary abelian p-group X of rank m, and that $X \cap G = 1$. In particular, $X \cap \langle y \rangle = 1$, so $W \stackrel{\text{def}}{=} X \cdot \langle y \rangle$ is a semi-direct product. Now, y^{p^a} is central in W, and

$$W/\langle y^{p^a} \rangle \cong X \rtimes \langle y \rangle / \langle y^{p^a} \rangle \cong X \rtimes C_{p^a},$$

where the cyclic group C_{p^a} acts regularly on the prescribed basis of X. It follows that the section $W/\langle y^{p^a} \rangle$ of $U(G)$ is isomorphic to $C_p \operatorname{wr} C_{p^a} \cong C_p \operatorname{wr} G'$, as required. □

REMARK. In general, we do not know whether Theorem A remains valid when $p = 2$. However, if G satisfies the extra condition $\gamma_3 \subseteq (G')^4$, then the conclusion follows, using arguments similar to those introduced here.

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